# Tunnelling through asymmetric parabolic potential barriers



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**COMMENT**

# Tunnelling through asymmetric parabolic potential barriers

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**Abstract.** An exact transmission coefficient for the asymmetric parabolic barrier potential, i.e. $V(x) = [V_1 - \frac{1}{2}m{\omega_1}^2x^2]\Theta(-x) + [V_2 - \frac{1}{2}m{\omega_2}^2x^2]\Theta(x)$, where $\Theta(x \leqslant 0) = 0$ and $\Theta(x > 0) = 1$, is rederived.

The asymmetric parabolic barrier (APB) potential is defined as

$$V(x) = [V_1 - \tfrac{1}{2}m{\omega_1}^2x^2]\Theta(-x) + [V_2 - \tfrac{1}{2}m{\omega_2}^2x^2]\Theta(x) \tag{1}$$

where $\Theta(x \leqslant 0) = 0$ and $\Theta(x > 0) = 1$. This potential has earlier [1] been referred to as an inverted biharmonic oscillator potential, and an analytic transmission coefficient has been proposed. This model barrier is found suitable to parametrize nuclear fission barriers [2] and also bears a pedagogical advantage [3].

It can be checked that the acclaimed transmission coefficient [1, 2] entails the following shortcomings: (i) it does not degenerate to $T(E)$ of the parabolic/harmonic barrier when $(\omega_1 = \omega_2)$; (ii) it does not yield the classical limit, i.e. $\mathop{\mathrm{Limit}}_{\hbar\to 0} T(E) = \Theta(E - V_0)$; (iii) it does not yield the high-energy limit, i.e. $T(E \to \infty) = 1$; and also (iv) it does not satisfy the unitarity, i.e. $T(E) \leqslant 1$. Although we found that a minor *ad-hoc* correction (squaring of the square bracket in equation (5)) enables $T(E)$ in [1, 2] to meet these *necessary* conditions successfully, yet the question of the correctness of $T(E)$ remained. Such thoughts have indeed set the ground for a rederivation of $T(E)$ for the biharmonic barrier. Thus, in this comment we intend to report the correct expression for $T(E)$ for the potential given in equation (1).

By defining $\alpha_1 = (V_1 - E)/\hbar\omega_1$, $\alpha_2 = (V_2 - E)/\hbar\omega_2$ and an asymmetry parameter, $\eta = \sqrt{\omega_2/\omega_1}$, we employ parabolic cylindrical functions [4], $E(\alpha, x)$, to find the transmission coefficient as

$$T(E) = \frac{4\eta}{|E'(\alpha_1, 0)E(\alpha_2, 0) + \eta E(\alpha_1, 0)E'(\alpha_2, 0)|^2}. \tag{2}$$

The function $E(a, 0)$ is analytically expressed as $E(a, 0) = 2^{-3/4}[k^{-1/2} + \mathrm{i}k^{1/2}]\sqrt{f(a)}$. Similarly, we have $E'(a, 0) = -2^{-1/4}[k^{-1/2} - \mathrm{i}k^{1/2}]/\sqrt{f(a)}$. The function $f(a)$ is defined as

$$f(a) = \left|\frac{\Gamma(1/4 + \mathrm{i}a/2)}{\Gamma(3/4 + \mathrm{i}a/2)}\right| \tag{3}$$

such that $f(-a) = f(a)$, $f(0) = 2.958\,71$, $f(\pm\infty) = 0$ and $k = \sqrt{1 + \mathrm{e}^{2\pi a}} - \mathrm{e}^{\pi a}$ [4]. The transmission coefficient, $T(E)$, finally simplifies to

$$T(E) = \frac{1}{\frac{1}{4}\sqrt{1 + \mathrm{e}^{2\pi\alpha_1}}\sqrt{1 + \mathrm{e}^{2\pi\alpha_2}}[\eta(f_1/f_2) + (1/\eta)(f_2/f_1)] + \frac{1}{2}[\mathrm{e}^{\pi\alpha_1}\,\mathrm{e}^{\pi\alpha_2} + 1]} \tag{4}$$

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where $f_1 = f(\alpha_1)$ and $f_2 = f(\alpha_2)$. Now let us rewrite the transmission coefficient of [1, 2] in a similar notation for the sake of comparison by denoting it as $T'(E)$:

$$T'(E) = \frac{\sqrt{\omega_1\omega_2}}{\frac{1}{4}\sqrt{1+\mathrm{e}^{2\pi\alpha_1}}\sqrt{1+\mathrm{e}^{2\pi\alpha_2}}[\sqrt{\omega_1}\sqrt{f_2/f_1}+\sqrt{\omega_2}\sqrt{f_1/f_2}]}. \tag{5}$$

Note the differences between (4) and (5).

Let us use $\underset{|y|\to\infty}{\mathrm{Limit}}|x+\mathrm{i}y| = \sqrt{2\pi}|y|^{x-1/2}\,\mathrm{e}^{-\pi|y|/2}$ to appreciate the large $\alpha$ behaviour of $f(\alpha)$. We obtain an important asymptotic expression as $f(\alpha) \sim |\alpha/2|^{-1/2}$. Using this we find two more interesting transmission coefficients: when $\omega_2 \to 0$, the APB potential presents a semi-infinite parabolic step barrier and we use the asymptotic value of $f(\alpha_2)$ in equation (4) to obtain

$$T^{\mathrm{step}}(E) = \frac{1}{\frac{1}{4}\sqrt{1+\mathrm{e}^{2\pi\alpha_1}}[f_1/\delta+\delta/f_1]+\frac{1}{2}}\Theta(E-V_2) \tag{6}$$

where $\delta = \sqrt{|V_2 - E|/2\hbar\omega_1}$. Note the step function above. Next, when $V_2 = 0$ and $\omega_2 \to 0$ the incident particle encounters half-a-parabolic barrier, since the potential for $x > 0$ is zero, and equation (6) yields

$$T^{\mathrm{half}}(E) = \frac{1}{\frac{1}{4}\sqrt{1+\mathrm{e}^{2\pi\alpha_1}}[f_1/\gamma+\gamma/f_1]+\frac{1}{2}} \tag{7}$$

where $\gamma = \sqrt{E/2\hbar\omega_1}$. Note the disappearance of the step function above.

## References


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